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Stone–Weierstrass Theorems in $C^*(X)$

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In this paper we consider some conditions for a given function $f \in C^*(X)$ to belong to the uniform closure of a subset $W \subset C^*(X)$. We start with a general theorem which admits a sensible improvement when W, or more generally its uniform closure cl(W), is a lattice. Also, we obtain an approximation result when W or cl(W) is a cone. From this result we can derive one by Blasco and Moltó for linear subspaces and one by Garrido and Montalvo for semi-affine lattices. Finally, using "multipliers," we extend to $C^*(X)$ some other known results for the compact case, such as the Nachbin theorem about the uniform closure of an A-module.

Key Words: uniform closure; Lebesgue sets; chain condition; multipliers.

1. INTRODUCTION

Let K be a compact Hausdorff space and let $W \subset C(K)$. It is well known that when W is a sublattice or a subalgebra of $C(K)$ the Kakutani–Stone theorem and the Weierstrass–Stone theorem characterize the uniform closure of W. Both theorems have many generalizations and extensions. Thus, Choquet and Deny [4] and Császár [5], have studied the uniform closure and in particular the uniform density for a lattice of $C(K)$ with some additional properties (convex sup-lattice, semi-affine lattice, etc.). In our opinion, the book *Weierstrass–Stone, the Theorem* [18] by J. B. Prolla is the most complete survey of this subject (compact spaces). It could be said that how points are separated by the functions in W originates from

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the different theorems. On the other hand, although the same arguments are not sufficient for the general case, there are some versions of these theorems for the noncompact case. Thus, Hewitt [11] in 1947 gave a uniform density theorem for algebras containing all the real constant functions of $C^*(X)$ (the set of all bounded and real-valued continuous functions over X). In the proof he considered separation of zero-sets in place of separation of points. Similar results have been obtained by Mrowka [15], Brosowsky and Deutsch [3], Blasco and Moltó [1] and Garrido–Montalvo [10].

In this paper, we present some new characterizations of the uniform closure of subfamilies $W \subset C^*(X)$ when X is a non-compact space. The first (Section 3) is very simple and general (no additional assumptions about W are needed). When either W or its uniform closure is a lattice we can simplify the assumption of the main result by means of the usual identification between the rings $C^*(X)$ and $C(\beta X)$. In particular, this allows us to complete the results obtained in [10] for lattices.

In Section 4 we modify the methods introduced in [9] for linear spaces of continuous functions to study more general families. Specifically, we relate the concept of S-separation of Lebesgue sets of bounded functions to the Chain Condition (defined in [7]) in a more general framework than the linear spaces. Indeed, in this section, the only assumption we make is that W or its uniform closure is a cone. As a consequence, we get a new characterization of $\mathrm{cl}(W)$. The theorem for semi-affine lattice given in [10] and the Blasco-Moltó [1] approximation theorem for linear spaces can be derived from those obtained here.

In Section 5 we get an extension to the bounded case of a version of the Stone–Weierstrass theorem given by Prolla [17] for arbitrary subsets of $C(K)$. We obtain first a theorem for arbitrary subsets of $C^*(X)$ and then a theorem for modules that can be regarded as an extension of Nachbin's theorem for the compact case (see [16]). This theorem gives, in particular, the Garrido–Montalvo [10] theorem for algebras and, hence, the above mentioned theorem of Hewitt.

2. PRELIMINARIES

Let X be a completely regular Hausdorff space. As usual, βX denotes the Stone–Čech compactification of X and $C^*(X)$ the family of all real-valued bounded continuous functions defined on X endowed with the uniform norm. Recall that βX is the only (up to homeomorphisms) Hausdorff compactification of X such that each $f \in C^*(X)$ admits a (unique) extension $f^\beta \in C^*(\beta X)$. Thus, if $W \subset C^*(X)$ and $W^\beta := \{f^\beta : f \in W\}$, then W and W^β, have similar algebraic properties. Moreover, $\mathrm{cl}_{C^*(\beta X)} W^\beta = (\mathrm{cl}_{C^*(X)} W)^\beta$, where $\mathrm{cl}_Y(A)$ denotes the closure of a set $A \subset Y$ with respect to the topology of Y. If no confusion is possible, we will omit the underlying space, that is $\mathrm{cl}(A) := \mathrm{cl}_Y(A)$. We will also use the notation

$$C(X, [0, 1]) := \{f \in C^*(X) : f(X) \subset [0, 1]\},$$

and $\mathbb{R}$ ($\mathbb{Z}$) for the family of all real (integer) numbers.

For $f \in C^*(X)$, the Lebesgue sets of f are defined by

$$L_a(f) := \{x \in X : f(x) \leqslant a\},$$
$$L^a(f) := \{x \in X : f(x) \geqslant a\}, \qquad (a \in \mathbb{R}).$$

Next, we list the separation properties which will be used. For a set $W \subset C^*(X)$ and a function $f\colon X \to \mathbb{R}$, we say that

1. *W S_1separates* the Lebesgue sets of f if for $a, b \in \mathbb{R}$, $a < b$, there exists $g \in W$ such that $0 \leqslant g \leqslant 1$, $g(L_a(f)) = \{0\}$ and $g(L^b(f)) = \{1\}$.

2. *W S-separates* the Lebesgue sets of f if for each $a, b \in \mathbb{R}$, $a < b$, and $\delta > 0$, there exists $g \in W$ such that, $0 \leqslant g \leqslant 1$, $g(L_a(f)) \subset [0, \delta]$ and $g(L^b(f)) \subset [1 - \delta, 1]$.

3. *W separates* the Lebesgue sets of f if for each $a, b \in \mathbb{R}$, $a < b$, there exists $g \in W$ such that $\mathrm{cl}_{\mathbb{R}}(g(L_a(f))) \cap \mathrm{cl}_{\mathbb{R}}(g(L^b(f))) = \varnothing$.

4. *W separates* the values of f if for each $x, y \in X$ such that $f(x) \neq f(y)$, there exists $g \in W$ such that $g(x) \neq g(y)$.

5. *W separates* points of X if, for each $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $g \in W$ such that $g(x_1) \neq g(x_2)$.

3. THE GENERAL CASE

We begin with the most general result in the paper.

THEOREM 3.1. *Let $W \subset C^*(X)$ be non-empty. Then $f \in \mathrm{cl}(W)$ if and only if, for every finite collection of real numbers $\alpha_1, \alpha_2, ..., \alpha_n$ and each $\varepsilon > 0$, there exists $g \in W$ such that*

$$|f(x) - \alpha_i| < \varepsilon \Rightarrow |g(x) - \alpha_i| < 2\varepsilon, \qquad (i = 1, ..., n). \tag{1}$$

Proof. If $f \in \mathrm{cl}(W)$ and $\varepsilon > 0$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$. Thus, if $\alpha \in \mathbb{R}$ and $|f(x) - \alpha| < \varepsilon$, then

$$|g(x) - \alpha| \leqslant |g(x) - f(x)| + |f(x) - \alpha| < 2\varepsilon.$$

Conversely, choose $\varepsilon > 0$ and let $\alpha_n = n\varepsilon$, $n \in \mathbb{Z}$. Because f is bounded, there exists a finite set $\Omega \subset \mathbb{Z}$ such that

$$X = \bigcup_{n \in \Omega} \mathscr{U}(f, \alpha_n, \varepsilon),$$

where $\mathscr{U}(f, \alpha_n, \varepsilon) := \{x : |f(x) - \alpha_n| < \varepsilon\}$. Let $g \in W$ be a function associated to the numbers $(\alpha_n)_{n \in \Omega}$ which satisfies (1). Then,

$$|f(x) - g(x)| \leqslant |f(x) - \alpha_n| + |g(x) - \alpha_n| < 3\varepsilon. \quad \blacksquare$$

If we have more information about the set W, how can the condition (1) be simplified? In this section we analyze the case in which $\mathrm{cl}(W)$ is a lattice (i.e., if g, $h \in \mathrm{cl}\, W$, then $g \vee h = \sup(g, h)$ and $g \wedge h = \inf(g, h)$ are in $\mathrm{cl}(W)$). We shall show below that if $\mathrm{cl}(W)$ is a lattice then we can take $n = 2$ in condition (1). To this aim, we shall apply to W^β the known theorem of Kakutani–Stone. Let us recall:

THEOREM 3.2 [12]. *Let X be a Hausdorff compact space, $W \subset C^*(X)$ a lattice and $f \in C^*(X)$ a fixed function. Then $f \in \mathrm{cl}(W)$ if and only if, for each $x_1, x_2 \in X$, $x_1 \neq x_2$, and $\varepsilon > 0$, there exists $g \in W$ such that*

$$|g(x_1) - f(x_1)| < \varepsilon, \qquad |g(x_2) - f(x_2)| < \varepsilon.$$

THEOREM 3.3. *Let X be a completely regular Hausdorff space, W a non-empty subset of $C^*(X)$ such that $\mathrm{cl}(W)$ is a lattice and $f \in C^*(X)$ a fixed function. The following assertions are equivalent*:

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each pair of real numbers a, b and every $\varepsilon > 0$, there exists $g \in W$ such that, for $x \in X$,*

$$|f(x) - a| < \varepsilon \Rightarrow |g(x) - a| < 2\varepsilon,$$

$$|f(x) - b| < \varepsilon \Rightarrow |g(x) - b| < 2\varepsilon.$$

Proof. Assertions (ii) follow easily from (i). To prove that (ii) implies (i), fix $p \neq q \in \beta X$, $\varepsilon > 0$ and $0 < \delta < \varepsilon/2$. From (ii) we get that there exists $g \in W$ such that

$$\{x \in X : |f(x) - f^\beta(p)| < \delta\} \subset \{x \in X : |g(x) - f^\beta(p)| < 2\delta\}$$

and

$$\{x \in X : |f(x) - f^\beta(q)| < \delta\} \subset \{x \in x : |g(x) - f^\beta(q)| < 2\delta\}.$$

Therefore,

$$p \in \{ y \in \beta X : |f^{\beta}(y) - f^{\beta}(p)| < \delta \} \subset \mathrm{cl}_{\beta X}\{ x \in X : |f(x) - f^{\beta}(p)| < \delta \}$$
$$\subset \mathrm{cl}_{\beta X}\{ x \in X : |g(x) - f^{\beta}(p)| < 2\delta \},$$

which means that $|g^{\beta}(p) - f^{\beta}(p)| \leqslant 2\delta \leqslant \varepsilon$. Likewise, we may prove that $|g^{\beta}(q) - f^{\beta}(q)| \leqslant \varepsilon$. Then the Kakutani–Stone theorem gives $f^{\beta} \in \mathrm{cl}\, W^{\beta}$, completing the proof. ▮

4. UNIFORM APPROXIMATION FOR CONES OF $C^*(X)$

In this section we consider the concept "chain condition" that was defined in [9]. Although here we only work with bounded functions, we now apply this condition to subsets $W \subset C^*(X)$ that are more general than linear spaces. Indeed, throughout this section we suppose W to be a subset of $C^*(X)$ such that $\mathrm{cl}(W)$ is a cone. Recall that a subset S in $C^*(X)$ is a cone when it satisfies the condition: for every $\lambda \geqslant 0$ and $g \in S$, $\lambda g \in S$.

DEFINITION 4.1. A set $W \subset C^*(X)$ satisfies the Chain Condition (CC) for a function $f \in C^*(X)$ if, for each finite collection of real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_{m+1}$ and for each $p \in \mathbb{Z}$, there exists $g \in W$ such that, for $n = 1, ..., m$,

$$x \in C_x := \{ x \in X : \alpha_{n-1} < f(x) < \alpha_{n+1} \} \Rightarrow |g(x) - (p+n)| < 2.$$

THEOREM 4.2. *Let $W \subset C^*(X)$ be such that* $\mathrm{cl}(W)$ *is a cone. Then, for a function $f \in C^*(X)$, the following assertions are equivalent*:

(i) *W satisfies* CC *for f.*

(ii) *$\varphi \circ f \in \mathrm{cl}(W)$ for every non-decreasing function $\varphi \in C(\mathbb{R})$.*

(iii) $\mathrm{cl}(W)$ *contains the lattice generated by the constant functions together with the functions of the form $f + c$, $c \in \mathbb{R}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\varphi \in C(\mathbb{R})$ be a non-decreasing function and set $c := \inf f(X)$ and $d := \sup f(X)$. Since φ is a non-decreasing continuous function, we have $\varphi(f(X) \subset \varphi([c, d])) = [\varphi(c), \varphi(d)]$. Fix $\varepsilon > 0$ and let $p \in \mathbb{Z}$ such that $(p-1)\,\varepsilon \leqslant \varphi(c) < p\varepsilon$. Denote by r the greatest natural number such that $(p + r - 2)\,\varepsilon \leqslant \varphi(d)$. Then, $r \geqslant 1$. Next, choose reals β_0, β_1, ..., β_{r+1} such that

$$\beta_0 < c,$$

$$\beta_1 = c,$$

$$\varphi(\beta_j) = (p+j-2)\,\varepsilon, \qquad \text{for} \quad 1 < j \leqslant r,$$

$$\beta_{r+1} > d.$$

It is clear that, for each $x \in X$, there exists $j (1 \leqslant j \leqslant r)$ such that:

$$(p+j-2)\,\varepsilon \leqslant \varphi(f(x)) < (p+j-1)\,\varepsilon.$$

Moreover, it is very simple to verify that, for $1 \leqslant j \leqslant r$,

$$\{x : (p+j-2)\,\varepsilon \leqslant \varphi(f(x)) < (p+j-1)\,\varepsilon\} \subset \{x : \beta_{j-1} < f(x) < \beta_{j+1}\}.$$

Denote by $C_j = \{x : \beta_{j-1} < f(x) < \beta_{j+1}\}$, $(1 \leqslant j \leqslant r)$. Then, if we take $g \in W$ such that

$$x \in C_j \Rightarrow |g(x) - (p+j)| < 2,$$

we obtain that, for each $x \in X$, $|\varepsilon g(x) - \varphi(f(x))| < 4\varepsilon$. Therefore, $\varphi \circ f \in \mathrm{cl}(W)$.

(ii) $\Rightarrow$ (iii) Let $L = \{\varphi \circ f : \varphi \in C(\mathbb{R})$ non-decreasing$\}$, then $L \subset \mathrm{cl}(W)$ and, as can easily be verified, L is a lattice which contains the constant functions and all functions of the form $f + c$, $c \in \mathbb{R}$.

(iii) $\Rightarrow$ (i) Firstly we shall prove that if we denote by S the lattice generated by the constant functions together with the functions of the form $f + c$, $c \in \mathbb{R}$, then $1/bf + c \in \mathrm{cl}(S)$ for every $b \geqslant 1$ and $c \in \mathbb{R}$.

Let $h = f + cb$ and fix α, β such that $\alpha \leqslant h(x) \leqslant \beta$ for every $x \in X$ and let $b \geqslant 1$, $n \in \mathbb{N}$ and $\varepsilon = (\beta - \alpha)/n$. Let

$$\psi(t) = \bigvee_{k=0}^{n-1} (bt - (b-1)(\alpha + k\varepsilon)) \wedge (\alpha + (k+1)\,\varepsilon).$$

It is easy to verify that if $t \in [\alpha, \beta]$ then $|\psi(t) - t| < \varepsilon$. Hence, $|(\psi \circ h)(x) - h(x)| < \varepsilon$ for every $x \in X$. Also it is clear that the function $g = 1/b\psi \circ h \in S$, and so $1/bh = 1/bf + c \in \mathrm{cl}(S)$.

Next, let $\alpha_0 < \alpha_1 < \cdots < \alpha_{m+1}$ be an arbitrary collection of real numbers and $p \in \mathbb{Z}$ and define the function $\varphi \in C([\alpha_0, \alpha_{m+1}])$ by

$$\varphi(t) = p + j + \frac{t - \alpha_j}{\alpha_{j+1} - \alpha_j} \qquad \text{if} \quad t \in [\alpha_j, \alpha_{j+1}], \qquad (j = 0, ..., m).$$

Notice that φ is a non-decreasing function such that, for $j=1, ..., m$, $\varphi([\alpha_j, \alpha_{j+1}]) \subset [p+j, p+j+1]$, so that

$$x \in C_j := \{y \in X : \alpha_{j-1} < f(y) < \alpha_{j+1}\} \Rightarrow |(\varphi \circ f(x)) - (p+j)| < 1. \quad (2)$$

Now, we shall have finished if we prove that $\varphi \circ f \in \mathrm{cl}(W)$. Write

$$a_j = \frac{1}{\alpha_{j+1} - \alpha_j}, \qquad c_j = p + j - \frac{\alpha_j}{\alpha_{j+1} - \alpha_j}, \qquad (j=0, ..., m),$$

and fix a real number b such that $b \geqslant a_j$ for every j. Thus, $b/a_j \geqslant 1$ and hence $a_j/bf + c_j/b \in \mathrm{cl}(S)$ which implies, since $\mathrm{cl}(S)$ is an lattice, that $1/b\varphi \circ f \in \mathrm{cl}(S)$. Finally, from $\mathrm{cl}(W)$ being a cone, it follows that $\varphi \circ f \in \mathrm{cl}(W)$. ∎

Remarks 4.3. (1) The hypothesis "$\mathrm{cl}(W)$ is a cone" is not needed for (i) to imply (ii) in Theorem 4.2. It can be replaced by the hypothesis "$\mathrm{cl}(W)$ is a divisible set", i.e., $g \in \mathrm{cl}(W) \Rightarrow 1/ng \in \mathrm{cl}(W)$ for every $n \in \mathbb{N}$ (or even more generally, $\lambda g \in \mathrm{cl}(W)$ for $\lambda \in \Gamma_1 \subset \mathbb{R}$ having 0 as accumulation point). Now, for (iii) to imply (i), we only need that if $g \in \mathrm{cl}(W)$ then $ng \in \mathrm{cl}(W)$ for every $n \in \mathbb{N}$ (or more generally $\lambda g \in \mathrm{cl}(W)$ for $\lambda \in \Gamma_2 \subset \mathbb{R}$ unbounded from above).

(2) The proof of Theorem 4.2 (see Eq. (2)) shows that the number 2 in the definition of the Chain Condition can be replace by any constant $C > 1$.

It follows from the above theorem that, if W is a set such that $\mathrm{cl}(W)$ is a cone or a divisible set, then the condition CC for a function $f \in C^*(X)$ is sufficient for that f and, more generally, all functions of the form $af + c$, $a \geqslant 0$, $c \in \mathbb{R}$ are in $\mathrm{cl}(W)$ (take in (ii) the non-decreasing function $\varphi(t) = at + c$). Moreover, as in [9], we can also give the following corollary:

COROLLARY 4.4. *Let $W \subset C^*(X)$ be such that* $\mathrm{cl}(W)$ *is a cone and W satisfies* CC *for all its functions. Then the following assertions are equivalent.*

(i) $f \in \mathrm{cl}(W)$.

(ii) *W satisfies* CC *for f.*

Moreover, W satisfies CC *for all its functions if and only if* $\mathrm{cl}(W)$ *also satisfies* CC.

Proof. We shall prove that, in this case, the Chain Condition is also necessary for $f \in \mathrm{cl}(W)$.

Fix $f \in \mathrm{cl}(W)$ and a non-decreasing function $\varphi \in C(\mathbb{R})$. If $\{g_n\}$ is a sequence of functions in W which converges uniformly to f, then the sequence $\{\varphi \circ g_n\}$ converges uniformly to $\varphi \circ f$ (notice that, f being a bounded function, φ is uniformly continuous on $f(X)$). Thus, we have

$\varphi \circ f \in \mathrm{cl}(W)$. The rest of the proof follows from the above equivalence by consideration of the remark 4.3(2). ∎

Notice that the condition "cl(W) is a lattice of $C^*(X)$ which contains all functions $ag + c$, $g \in W$, $a \geqslant 0$, $c \in \mathbb{R}$" is a sufficient condition for W satisfy CC for all its functions. But the following example shows that is not necessary.

EXAMPLE 4.5 Let W be the set of the all monotone functions of $C^*(\mathbb{R})$. It is clear that although W is a uniformly closed cone of $C^*(\mathbb{R})$ which satisfies CC for every $g \in W$, it is not a lattice.

It was proved in [8] that, when W is a linear subspace of $C^*(X)$, W satisfies CC for a $f \in C^*(X)$ and W S-separates the Lebesgue sets of f are equivalent statements. But, as the next example shows, the same conclusion can not be obtained from the single assumption of cl(W) being a cone:

EXAMPLE 4.6.[2] Let W be the family of all functions of $f \in C^*(\mathbb{N})$ such that the set $f(\mathbb{N})$ consists of no more than two real numbers. As can be easily verified, W is a uniform closed cone which contains the constant functions. Thus, taking into account Corollary 4.4, W satisfies CC for a function $f \in C^*(\mathbb{N})$ if and only if $f \in W$. However, W S_1-separates the Lebesgue sets of every function in $C^*(\mathbb{N})$.

In the next proposition we consider some cases which S-separation of Lebesgue sets and the Chain Condition are equivalent.

PROPOSITION 4.7. *Let $W \subset C^*(X)$ be such that* cl(W) *is either a convex cone containing the constant functions or a lattice containing the functions of the form $ag + c$, $g \in W$, $a \geqslant 0$, $c \in \mathbb{R}$. Then, for a function $f \in C^*(X)$, the following assertions are equivalent*:

(i) *W satisfies* CC *for f.*

(ii) *W S-separates the Lebesgue sets of f.*

Proof. (i) implies (ii). Let us suppose that W satisfies CC for f and fix $a < b$ in $\mathbb{R}$. Choose $0 < \delta < 1$ and $\varepsilon < \delta/2$. Consider a non-decreasing function $\varphi \in C^*(\mathbb{R})$ such that, for $t \leqslant a$, $\varphi(t) = \varepsilon$ and, for $t \geqslant b$, $\varphi(t) = 1 - \varepsilon$. Then, from Theorem 4.2 it follows that $\varphi \circ f \in \mathrm{cl}(W)$. Let $g \in W$ be such that $|g - \varphi \circ f| < \varepsilon$. Then, taking into account that

$$x \in L_a(f) \Rightarrow (\varphi \circ f)(x) = \varepsilon,$$

$$x \in L^b(f) \Rightarrow (\varphi \circ f)(x) = 1 - \varepsilon,$$

[2] This example was supplied by I. Garrido.

we can verify that

$$0 \leqslant g \leqslant 1, \qquad g(L_a f) \subset [0, \delta], \qquad g(L^b f) \subset [1-\delta, 1].$$

(Notice that, in this part of the proof, no additional assumptions about W are needed.)

(ii) implies (i). First, let us assume that $\mathrm{cl}(W)$ is a convex cone and that W S-separates the Lebesgue sets of f. Fix a finite collection of real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_{m+1}$ and $\delta < 1/m$. For each $n = 1, ..., m$, choose a function of $g_n \in W$, $0 \leqslant g_n \leqslant 1$, such that

$$g_n(L_{\alpha_{n-1}}(f)) \subset [0, \delta] \qquad g_n(L^{\alpha_n}(f)) \subset [1-\delta, 1],$$

and define $g = \sum_{n=1}^m g_n$. Since $\mathrm{cl}(W)$ is convex, $g \in \mathrm{cl}(W)$, and a straightforward calculation shows (see [8]) that

$$x \in C_n = \{x : \alpha_{n-1} < f(x) < \alpha_{n+1}\} \Rightarrow |g(x) - n| < 1 + m\delta.$$

From this equation, we infer that W satisfies CC for f. In fact, if $p \in \mathbb{Z}$, then $g + p$ belongs to $\mathrm{cl}(W)$. Thus, taking $h \in W$ such that $|g(x) + p - h(x)| < \delta$ for every $x \in X$, one has that

$$x \in C_n \Rightarrow |h(x) - (p+n)| < 2.$$

Notice that for the above proof to hold, it suffices that $\mathrm{cl}(W)$ contains the constant functions and that $f, g \in W$ implies $f + g \in \mathrm{cl}(W)$.

Finally, let us assume that $\mathrm{cl}(W)$ is a lattice which contains all the functions of the form $ag + c$ ($g \in W$, $a \geqslant 0$, $c \in \mathbb{R}$). If W S-separates the Lebesgue sets, then $\mathrm{cl}(W)$ S^1-separates the Lebesgue sets. In fact, if $0 \leqslant g \leqslant 1$ satisfies

$$g(L_a f) \subset [0, \delta], \qquad g(L^b f) \subset [1-\delta, 1], \qquad 0 < \delta < 1/2,$$

then the function $h = 1/(1-2\delta)((g-\delta) \vee 0) \wedge 1$ (which, taking into account our assumptions, is in $\mathrm{cl}(W)$) satisfies the equations $h(L_a f) = 0$ and $h(L^b f) = 1$. Fix a finite collection of real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_{m+1}$ and consider, for each $n = 1, ..., m-1$, a function $g_n \in \mathrm{cl}(W)$, $0 \leqslant g_n \leqslant 1$, such that

$$g_n(L_{\alpha_{n-1}}(f)) = 0, \qquad g_n(L^{\alpha_n}(f)) = 1.$$

Define $g := \bigvee_{n=1}^m n g_n$. It is easy to prove that, if $x \in C_n$, then $|g(x) - n| \leqslant 1$. Therefore, as before, we have that W satisfies CC for f. ∎

Remark 4.8. Notice that the uniform closed cone W in Example 4.5 is neither convex nor a lattice, but it is easy to verify that W satisfies CC for a function $f \in C^*(X)$ if and only if W S-separates the Lebesgue sets of f.

On the other hand, the example 4.6 shows that (i) and (ii) may not be equivalent, even if W satisfies CC for every function $g \in W$.

THEOREM 4.9. *Let $W \subset C^*(X)$ be such that either* cl(W) *is a lattice which contains all functions of the form $ag + c$, $g \in W$, $a \geqslant 0$, $c \in \mathbb{R}$, or* cl(W) *is a convex cone which satisfies* CC *for its functions. Then the following assertions are equivalent.*

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each $a < b$ and $\varepsilon > 0$, there exists $g \in W$ such that*

$$|g(x) - a| < \varepsilon \quad \textit{if } x \in L_a(f), \qquad |g(x) - b| < \varepsilon \quad \textit{if } x \in L^b(f).$$

(iii) *W satisfies* CC *for f.*

(iv) *W S-separates the Lebesgue sets of f.*

(v) *For each $a < b$, there exists $g \in W$ such that*

$$\sup\{g(x) : x \in L_a(f)\} < \inf\{g(x) : x \in L^b(f)\}.$$

Proof. We shall show that (i) ⇒ (v) ⇒ (ii) ⇒ (iv) ⇒ (iii) ⇒ (i). It is clear that (i) ⇒ (v). To prove that (v) ⇒ (ii) let $a < b$ and choose $g \in W$ such that

$$\alpha = \sup\{g(x) : x \in L_a(f)\} < \inf\{g(x) : x \in L^b(f)\} = \beta.$$

Then, take a non-decreasing function $\varphi \in C^*(\mathbb{R})$ satisfying $\varphi(t) = a - \varepsilon$ if $t \leqslant \alpha$ and $\varphi(t) = b + \varepsilon$ if $t \geqslant \beta$. The map $\varphi \circ g \in \mathrm{cl}(W)$ because W satisfies CC for all its functions. Moreover, $(\varphi \circ g)(x) = \varphi(\alpha) = a - \varepsilon$ if $x \in L_a(f)$ and $(\varphi \circ g)(x) = b + \varepsilon$ if $x \in L^b(f)$. Now, choose $h \in W$ such that $|\varphi \circ g - h| < \varepsilon$. Then, $|h(x) - a| \leqslant 2\varepsilon$ if $x \in L_a(f)$ and $|h(x) - b| \leqslant 2\varepsilon$ if $x \in L^b(f)$. To prove that (ii) ⇒ (iv) we shall use a similar argument. Let $g \in W$ be such that

$$|g(x) - a| < \varepsilon \quad \text{if } x \in L_a(f), \qquad |g(x) - b| < \varepsilon \quad \text{if } x \in L^b(f).$$

For each $0 < \delta < 1$ let δ_1 be such that $0 < \delta_1 < \delta$ and $2\delta_1 < 1 - 2\delta_1$. Take $\varphi \in C^*(\mathbb{R})$ be a non-decreasing function such that $\varphi(t) = \delta_1$ if $t \leqslant a + \varepsilon$ and $\varphi(t) = 1 - \delta_1$ if $t \geqslant b - \varepsilon$. Then the map $\varphi \circ g \in \mathrm{cl}(W)$ and, taking $h \in W$ with $|\varphi \circ g - h| \leqslant \delta_1$, we get

$$0 \leqslant h \leqslant 1, \qquad h(L_a(f)) \subset [0, \delta), \qquad h(L^b(f)) \subset (1 - \delta, 1].$$

Finally, from Proposition 4.7 it follows that (iv) is equivalent to (iii) and (i). ∎

Remark 4.10. It is known that if $W \subset C^*(X)$ then a necessary condition for a function f to belong to cl(W) is that W separates the Lebesgue sets of f. This condition is not sufficient, even if W is as in the above theorem. For example, if W is the set of continuous and non-decreasing real functions over $[0, 1]$, it is clear that W is a uniform closed lattice which contains all functions of the form $ag + c$, $g \in W$, $a \geqslant 0$, $c \in \mathbb{R}$. Since the identity function belongs to W, it is obvious that W separates the Lebesgue sets of every function in $C[0, 1]$, and nevertheless $\mathrm{cl}(W) = W$.

But if we suppose now that $\mathrm{cl}(W)$ is an affine lattice, i.e., a lattice which contains all the functions of the form $ag + c$, a, $c \in \mathbb{R}$, $g \in \mathrm{cl}(W)$ (including when a is not a positive real number), then the separation of Lebesgue sets of f is another equivalent for $f \in \mathrm{cl}(W)$:

COROLLARY 4.11. *Let* $W \subset C^*(X)$ *be such that* $\mathrm{cl}(W)$ *is an affine lattice. Then the conditions of Theorem* 4.9, *for a function* $f \in C^*(X)$, *are equivalent to*:

(vi) *W separates the Lebesgue sets of f.*

Proof. First, notice that, taking into account that $\mathrm{cl}(W)$ is an affine lattice, if $g \in W$ then $\varphi \circ g \in \mathrm{cl}(W)$ for every $\varphi \in C^*(\mathbb{R})$. In fact, if $K = \mathrm{cl}_{\mathbb{R}}(g(X)$ then $\mathscr{L} = \{\varphi \in C(K) : \varphi \circ g \in \mathrm{cl}(W)\}$ is a uniform closed affine lattice of continuous functions on the compact space K which separates points. Thus, from Kakutani–Stone theorem (see, for instance, [6]) it follows that $\mathscr{L} = C(K)$.

Therefore, the condition "W separates the Lebesgue sets of f" implies that "$\mathrm{cl}(W)$ S_1-separates the Lebesgue sets of f" and hence, from the above theorem, that $f \in cl(W)$. ∎

We have seen in the preceding results that, when $\mathrm{cl}(W)$ is an affine lattice, there exist many different ways to know if a function f belongs to $\mathrm{cl}(W)$. Thus, we are interested in recognizing when the uniform closure of a set $W \subset C^*(X)$ is an affine lattice. This is the sense of the next result, which includes, in particular, another of Blasco and Moltó [1]. We give one prior definition: Let W be a subset of $C^*(X)$. We shall say that W is a *semi-affine lattice* if W is a lattice and if $f \in W$ then $f + c \in W$ and $\mu f \in W$ for every $c \in \mathbb{R}$ and $\mu \in \Gamma$ where Γ is a set of real numbers containing 0 and unbounded both from above and from below. Notice that a semi-affine lattice must contain the constant functions, and that a sufficient condition for a lattice W containing the constant functions to be a semi-affine lattice is that if f, $g \in W$ then $f - g \in W$, i.e., that W is a *substractive lattice*.

COROLLARY 4.12. *Let* $W \subset C^*(X)$. *Then, for* $\mathrm{cl}(W)$ *to be an affine lattice, it is sufficient that either of the following conditions is satisfied,*

(i) *W is a divisible group which S-separates the Lebesgue sets of every* $g \in W$.

(ii) *W is a semi-affine lattice.*

In particular, conditions (i)–(vi), *defined above, are equivalent in both cases.*

Proof. Assume that W is a divisible group which S-separates the Lebesgue sets of every $g \in W$. From Proposition 4.7, it follows that W satisfies CC for every $g \in W$. Now, taking into account Remark 4.3(1), we can apply Theorem 4.2 to W and hence it follows that for every $f \in \mathrm{cl}(W)$ one has $af + c$, $-af + c$, $f \vee 0 \in \mathrm{cl}(W)$ for every a, $c \in \mathbb{R}$ (notice that $\mathrm{cl}(W)$ is a group). Also, we can write $f \vee g = g + (f - g) \vee 0$, and hence from the above one has that $\mathrm{cl}(W)$ is an affine lattice.

For W a semi-affine lattice, we know that if $f \in \mathrm{cl}(W)$ and $0 \leqslant a \leqslant 1$ then $af \in \mathrm{cl}(W)$ (see the last part in the proof of Theorem 4.2). For $a > 1$ let $b \in \Gamma$ such that $b > a$. Then $(a/b) f \in \mathrm{cl}(W)$, and so $b(a/b) f = af \in \mathrm{cl}(W)$. ∎

Remark 4.13. In [10] Garrido–Montalvo showed that if W is a lattice containing the real constant functions then condition (i) is equivalent to condition (ii) and that if W is a semi-affine lattice then conditions (iv)–(vi) are equivalent to (i).

5. MULTIPLIERS: UNIFORM APPROXIMATION FOR MODULES OF $C^*(X)$

L. Nachbin presented in [16] a version of the Stone–Weierstrass theorem for modules of continuous functions on a compact Hausdorff space K. J. B. Prolla, based upon some ideas of S. Machado in [14], extended the results of Nachbin to arbitrary sets of $C(K)$ (see [17]). In this section, we shall use these results to obtain new forms of characterizing the uniform closure of a subset of $C^*(X)$ for X a completely regular, in general, non-compact space.

Let W be a non-empty subset of $C^*(X)$. A function $\varphi \in C(X; [0, 1])$ is said to be a *multiplier* of W if, whenever g, h are functions in W, one has that $\varphi g + (1 - \varphi) h \in W$. The family of all multipliers of W is denoted by $M(W)$. Here, for $x \in X$, we use the notations

$$[x] = \{ y \in X : \varphi(y) = \varphi(x), \forall \varphi \in M(W) \}$$

and

$$W_{[x]} = \{ g_{[x]} \text{ (restriction of } g \text{ to } [x]) : g \in W \}.$$

Let us recall the result of J. B. Prolla.

THEOREM 5.1 [17, Prolla]. *Let K be a compact Hausdorff space, $W \subset C(K)$ and $f \in C(K)$. Then the following conditions are equivalent.*

1. $f \in \mathrm{cl}(W)$.
2. *For each* $x \in K$, $f_{[x]} \in \mathrm{cl}(W_{[x]})$.

COROLLARY 5.2. *Let K be a compact Hausdorff space, $W \subset C(K)$ and $f \in C(K)$. If $M(W)$ separates the values of every function in $W \cup \{f\}$, then the following conditions are equivalent.*

1. $f \in \mathrm{cl}(W)$.
2. *For every* $x \in K$, $f(x) \in \mathrm{cl}\{g(x) : g \in W\} := \mathrm{cl}(W(x))$.

Proof. It follows directly from Theorem 5.1. Notice that the condition imposed on the multipliers is equivalent to supposing that, for $x \in K$, f and all functions in W are constant functions on $[x]$. ∎

Next, X is, in general, a non-compact space and, as is usual, we identify $C^*(X)$ with $C(\beta X)$. We use the identity $M(W^\beta) = M(W)^\beta$. Likewise, we shall apply the following standard lemma:

LEMMA 5.3. *Let $W \subset C^*(X)$ and $f \in C^*(X)$ be such that W separates the Lebesgue sets of f. Then W^β separates the values of f^β.*

Proof. See [10, Theorem 10].

THEOREM 5.4. *Let $W \subset C^*(X)$ and suppose that $M(W)$ separates the Lebesgue sets of the functions in W. Then, for a function $f \in C^*(X)$, the following assertions are equivalent*:

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each $\varepsilon > 0$ and for each pair a, b of real numbers, there exists a function $g \in W$ such that*:

$$|f(x) - a| < \varepsilon \Rightarrow |g(x) - a| < 2\varepsilon,$$

$$|f(x) - b| < \varepsilon \Rightarrow |g(x) - b| < 2\varepsilon.$$

(iii) *$M(W)$ separates the Lebesgue sets of f, and for each $\varepsilon > 0$ and $a \in \mathbb{R}$, there exists a function $g \in W$ such that*

$$|f(x) - a| < \varepsilon \Rightarrow |g(x) - a| < 2\varepsilon.$$

Proof. We will apply Corollary 5.2 to W^β. I.e., we will prove that it follows from condition (ii) or (iii) that $M(W^\beta)$ separates the values of the functions in $W^\beta \cup \{f^\beta\}$, and that, for every point $p \in \beta X$, $f^\beta(p) \in \mathrm{cl}(W^\beta)(p)$.

From Lemma 5.3, if $g \in W$ and $p, q \in \beta X$ are such that $g^\beta(p) \neq g^\beta(q)$, then there exists a function $\varphi \in M(W)$ such that $\varphi^\beta(p) \neq \varphi^\beta(q)$.

Let us suppose that (ii) holds. Then, as in the proof of Theorem 3.3, it follows that, if $p \neq q \in \beta X$ and $\varepsilon > 0$, there exists $g \in W$ such that

$$|g^\beta(p) - f^\beta(p)| \leqslant 2\varepsilon, \qquad |g^\beta(q) - f^\beta(q)| \leqslant 2\varepsilon. \tag{3}$$

Now, it is clear that, for each $p \in \beta X$, $f^\beta(p) \in \mathrm{cl}\ W^\beta(p)$, and that $M(W^\beta)$ separates the values of the function f^β. In fact, if $f^\beta(p) < f^\beta(q)$ and we choose $0 < \varepsilon < 1/4(f^\beta(q) - f^\beta(p))$, then the inequalities (3) imply that $g^\beta(p) \neq g^\beta(q)$. Since $M(W^\beta)$ separates the values of the function g^β, we have the desired property, i.e., a function $\varphi \in M(W)$ such that $\varphi^\beta(p) \neq \varphi^\beta(q)$.

We have also proved in the above that condition (iii) implies $f \in \mathrm{cl}(W)$.

To finish the proof, we only need to show that the conditions (ii) and (iii) are necessary for $f \in \mathrm{cl}(W)$. Choose $a < b$, $\varepsilon < (b-a)/2$ and $g \in W$ such that $|f - g| < \varepsilon$. Then, it is obvious that (ii) holds, and to show (iii) it suffices to see that $M(W)$ separates the Lebesgue sets of f. But, if $\varphi \in M(W)$ separates the sets $L_{a+\varepsilon}(g)$ and $L^{b-\varepsilon}(g)$, this function separates the sets $L_a(f)$ and $L^b(f)$. Thus, conditions (ii) and (iii) are verified. ∎

COROLLARY 5.5. *Let $W \subset C^*(X)$ be such that $M(W)$ separates the Lebesgue sets of all functions in W and, for each $\lambda \in \mathbb{R}$, $\lambda W \subset W$. For a function $f \in C^*(X)$, the following assertions are equivalent.*

(i) $f \in \mathrm{cl}(W)$.

(ii) *$M(W)$ separates the Lebesgue sets of f, and, for each $\varepsilon > 0$, there exists a function $g \in W$ such that*

$$L^{2\varepsilon}(|f|) \subset L^{\varepsilon}(|g|) \tag{4}$$

Proof. It is easy to show that the new condition (4) is necessary in order to have $f \in \mathrm{cl}(W)$. Hence (see Corollary 5.2), we only need to prove that such a condition implies $f^\beta(p) \in \mathrm{cl}\ W^\beta(p)$, whenever $p \in \beta X$. But, taking into account the assumption about W ($\lambda W \subset W$), it is sufficient to show that, if $f^\beta(p) \neq 0$, there exists $g \in W$ such hat $g^\beta(p) \neq 0$, i.e., W^β does not vanish on the points where f^β does not vanish.

Let us suppose $f^\beta(p) \neq 0$ and let $\varepsilon = 1/4\,|f^\beta(p)|$. Then, if $g \in W$ satisfies the condition $L^{2\varepsilon}(|f|) \subset L^{\varepsilon}(|g|)$, we obtain (as in the proof of Theorem 10 in [10]), that $|g^\beta(p)| \geqslant 1/4\,|f^\beta(p)| > 0$. ∎

In the next corollary, we present our version of the Stone–Weierstrass theorem for modules of $C^*(X)$. Let A be a subalgebra of $C^*(X)$ (i.e., A is a linear subspace such that if $f, g \in A$ then $f \cdot g \in A$). A linear subspace $W \subset C^*(X)$ is called an A-module if $\varphi g \in W$, for every $\varphi \in A$ and $g \in W$.

COROLLARY 5.6. *Let W be an A-module of $C^*(X)$ and $f \in C^*(X)$ and assume that A separates the Lebesgue sets of all functions in W. Then $f \in \mathrm{cl}(W)$ if and only if the following two conditions are satisfied*:

(i) *A separates the Lebesgue sets of f.*

(ii) *For each $\varepsilon > 0$ there exists $g \in W$ such that $L^{2\varepsilon}(|f|) \subset L^{\varepsilon}(|g|)$.*

Proof. First, we prove that the condition "A separates the Lebesgue sets of a function $h \in C^*(X)$" implies that $M(W)$ also separates it. In fact, because W in an A-module, it follows that if we denote by B the set

$$B = \{\varphi \in C^*(X) : \varphi g \in W, \forall g \in W\},$$

then $A \subset B$ and hence B also separates the Lebesgue sets of h. Moreover, since W is a linear space, it is easy to see that

$$M(W) = \{\varphi \in B : 0 \leqslant \varphi \leqslant 1\}.$$

Therefore, we must prove that the functions $\varphi \in B$ such that $0 \leqslant \varphi \leqslant 1$ also separate the Lebesgue sets of h. But, since B is a subalgebra of $C^*(X)$ which contains the constant functions, then $\mathrm{cl}(B)$ is a linear lattice (Lebesgue's lemma [13]) which contains the constant functions. Now, from Corollary 4.11 B separates if and only if it S-separates.

From the above and the Corollary 5.5, it follows at once that the conditions are sufficient. They are also necessary because, if A separates the Lebesgue sets of every function in W, then A separates the Lebesgue sets of every function in $\mathrm{cl}(W)$. ▮

COROLLARY 5.7 [10, Garrido–Montalvo]. *Let W be a subalgebra of $C^*(X)$ and $f \in C^*(X)$. Then $f \in \mathrm{cl}(W)$ if and only if the following two conditions are satisfied*:

(i) *W separates the Lebesgue sets of f.*

(ii) *For each $\varepsilon > 0$ there exists $g \in W$ such that $L^{2\varepsilon}(|f|) \subset L^{\varepsilon}(|g|)$.*

Proof. If W is an algebra, then

$$W \subset A = \{\varphi \in C^*(X) : \varphi g \in W, \forall g \in W\}.$$

Therefore, since W separates the Lebesgue sets of all its functions, A separates the Lebesgue sets of W. Finally, the result follows from Corollary 5.6. ∎

Remark 5.8. The usual condition throughout this section, "$M(W)$ separates the Lebesgue sets of all the functions in W", is very restricted. In fact, it is possible to find examples for $W \subset C^*(X)$ in which only the general theorem (Theorem 3.1) applies.

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